

GI -graphs and their groups

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Abstract

The class of generalized Petersen graphs was introduced by Coxeter in the 1950s. Frucht, Graver and Watkins determined the automorphism groups of generalized Petersen graphs in 1971, and much later, Nedela and Škovič and (independently) Lovrečič-Saražin characterised those which are Cayley graphs. In this paper we extend the class of generalized Petersen graphs to a class of GI -graphs. For any positive integer n and any sequence j_0, j_1, \dots, j_{t-1} of integers mod n , the GI -graph $GI(n; j_0, j_1, \dots, j_{t-1})$ is a $(t+1)$ -valent graph on the vertex set $\mathbb{Z}_t \times \mathbb{Z}_n$, with edges of two kinds:

- an edge from (s, v) to (s', v) , for all distinct $s, s' \in \mathbb{Z}_t$ and all $v \in \mathbb{Z}_n$,
- edges from (s, v) to $(s, v + j_s)$ and $(s, v - j_s)$, for all $s \in \mathbb{Z}_t$ and $v \in \mathbb{Z}_n$.

By classifying different kinds of automorphisms, we describe the automorphism group of each GI -graph, and determine which GI -graphs are vertex-transitive and which are Cayley graphs. A GI -graph can be edge-transitive only when $t \leq 3$, or equivalently, for valence at most 4. We present a unit-distance drawing of a remarkable $GI(7; 1, 2, 3)$.

Keywords: GI -graph, generalized Petersen graph, vertex-transitive graph, edge-transitive graph, circulant graph, automorphism group, wreath product, unit-distance graph.

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1 Introduction

Trivalent graphs (also known as cubic graphs) form an extensively studied class of graphs. Among them, the Petersen graph is one of the most important finite graphs, constructible in many ways, and is a minimal counter-example for many conjectures in graph theory. The Petersen graph is the initial member of a family of graphs $G(n, k)$, known today as *Generalized Petersen graphs*, which have similar constructions. Generalized Petersen graphs were first introduced by Coxeter [3] in 1950, and were named in 1969 by Watkins [19].

A standard visualization of a generalized Petersen graph consists of two types of vertices: half of them belong to an outer rim, and the other half belong to an inner rim; and there are three types of edges: those in the outer rim, those in the inner rim, and the ‘spokes’, which form a 1-factor between the inner rim and the outer rim. The outer rim is always a cycle, while the inner rim may consist of several isomorphic cycles. A generalized Petersen graph $G(n, k)$ is given by two parameters n and k , where n is the number of vertices in each rim, and k is the ‘span’ of the inner rim (which is the distance on the outer rim between the neighbours of two adjacent vertices on the inner rim).

The family $G(n, k)$ contains some very important graphs. Among others of particular interest are the n -prism $G(n, 1)$, the Dürer graph $G(6, 2)$, the Möbius-Kantor graph $G(8, 3)$, the dodecahedron $G(10, 2)$, the Desargues graph $G(10, 3)$, the Nauru graph $G(12, 5)$, and of course the Petersen graph itself, which is $G(5, 2)$.

Generalized Petersen graphs possess a number of interesting properties. For example, $G(n, k)$ is vertex-transitive if and only if either $n = 10$ and $k = 2$, or $k^2 \equiv \pm 1 \pmod n$ [6], and a Cayley graph if and only if $k^2 \equiv 1 \pmod n$ [14, 17], and arc-transitive only in the following seven cases: $(n, k) = (4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5)$ or $(24, 5)$ [6].

If we want to maintain the symmetry between the two rims, then another parameter has to be introduced, allowing the span on the outer rim to be different from 1. This gives the definition of an *I-graph*.

The family of *I-graphs* was introduced in 1988 in the Foster Census [4]. For some time this family failed to attract the attention of many researchers, possibly due to the fact that among all *I-graphs*, the only ones that are vertex-transitive are the generalized Petersen graphs [1, 15]. Still, necessary and sufficient conditions for testing whether or not two *I-graphs* are isomorphic were determined in [1, 11], and these were used to enumerate all *I-graphs* in [18]. Also in [11] it was shown that all generalized Petersen graphs are unit-distance graphs, by representing them as isomorphic *I-graphs*. Furthermore, in [1] it was shown that automorphism group of a connected *I-graph* $I(n, j, k)$ that is not a generalized Petersen graph is either dihedral or a group with presentation

$$\Gamma = \langle \rho, \tau, \varphi \mid \rho^n = \tau^2 = \varphi^2 = 1, \rho\tau\rho = \tau, \varphi\tau\varphi = \tau, \varphi\rho\varphi = \rho^a \rangle$$

for some $a \in \mathbb{Z}_n$, and that among all *I-graphs*, only the generalized Petersen graphs can be vertex-transitive or edge-transitive.

In this paper we further generalize both of these families of graphs, and call them *generalized I-graphs*, or simply *GI-graphs*. We determine the group of automorphisms of any *GI-graph*. Moreover, we completely characterize the edge-transitive, vertex-transitive and Cayley graphs, among the class of *GI-graphs*.

At the end of the paper we briefly discuss the problem of unit-distance realizations of *GI-graphs*. This problem has been solved for *I-graphs* in [10]. We found a remarkable new example of a 4-valent unit-distance graph, namely $GI(7; 1, 2, 3)$, which is a Cayley graph on 21 vertices for the group $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

Let us note that ours is not the only possible generalization. For instance, see [13] for another approach, which is not much different from ours. The basic difference is that our approach uses complete graphs, while the approach by Lovrečić-Saražin, Pacco and Previtali in [13] uses cycles; their construction coincides with ours for $t \leq 3$, but not for larger t .

We acknowledge the use of MAGMA [2] in constructing and analysing examples of GI -graphs, and helping us to see patterns and test conjectures that led to many of the observations made and proved in this paper.

2 Definition of GI -graphs and their properties

For positive integers n and t with $n \geq 3$, let $(j_0, j_1, \dots, j_{t-1})$ be any sequence of integers such that $0 < j_k < n$ and $j_k \neq n/2$, for $0 \leq k < t$.

Then we define $GI(n; j_0, j_1, \dots, j_{t-1})$ to be the graph with vertex set $\mathbb{Z}_t \times \mathbb{Z}_n$, and with edges of two types:

- (a) an edge from (s, v) to (s', v) , for all distinct $s, s' \in \mathbb{Z}_t$ and all $v \in \mathbb{Z}_n$,
- (b) edges from (s, v) to $(s, v + j_s)$ and $(s, v - j_s)$, for all $s \in \mathbb{Z}_t$ and all $v \in \mathbb{Z}_n$.

This definition gives us an infinite family of graphs, which we call *GI-graphs*.

The graph $GI(n; j_0, j_1, \dots, j_{t-1})$ has nt vertices, and is regular of valence $(t-1)+2 = t+1$. Edges of type (a) are called the *spoke edges*, while those of type (b) are called the *layer edges*. Also for each $s \in \mathbb{Z}_t$ the set $L_s = \{(s, v) : v \in \mathbb{Z}_n\}$ is called a *layer*, and for each $v \in \mathbb{Z}_n$ the set $S_v = \{(s, v) : s \in \mathbb{Z}_t\}$ is called a *spoke*. We observe that the induced subgraph on each spoke is a complete graph K_t of order t . On the other hand, the induced subgraph on the layer L_s is a union of d_s cycles of length n/d_s , where $d_s = \gcd(n, j_s)$.

In the case $t = 1$, the graph $GI(n; j_0)$ is simply a union of disjoint isomorphic cycles of length $n/\gcd(n, j_0)$. In the case $t = 2$, we have *I*-graphs; for example, $GI(n; 1, j)$ is a generalized Petersen graph, for every j , and in particular, $GI(5; 1, 2)$ is the dodecahedral graph (the 1-skeleton of a dodecahedron). Some other examples are illustrated in Figure 1.

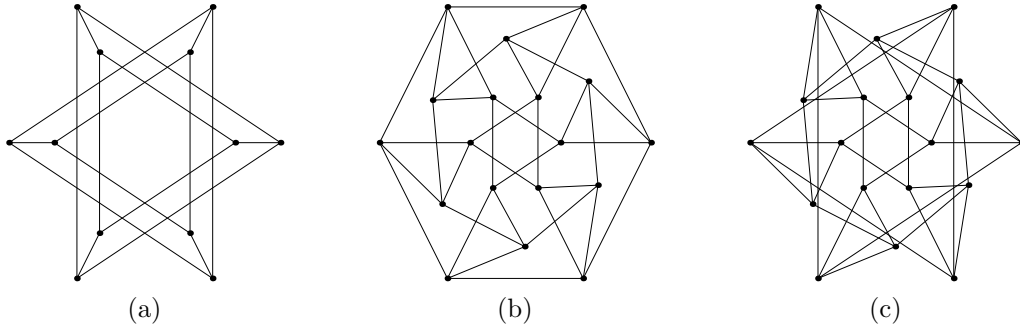


Figure 1: GI -graphs $GI(6; 2, 2)$, $GI(6; 1, 1, 2)$, and $GI(6; 2, 1, 2)$.

Note that taking $j'_k = \pm j_k$ for all k gives a GI -graph $GI(n; j'_0, j'_1, \dots, j'_{t-1})$ that is exactly the same as $GI(n; j_0, j_1, \dots, j_{t-1})$. Similarly, any permutation of j_0, j_1, \dots, j_{t-1} gives a GI -graph isomorphic to $GI(n; j_0, j_1, \dots, j_{t-1})$. Therefore we will usually assume that $0 < j_k < n/2$ for all k , and that $j_0 \leq j_1 \leq \dots \leq j_{t-1}$. In this case, we say that the GI -graph $GI(n; j_0, j_1, \dots, j_{t-1})$ is in *standard form*.

The following theorem gives a partial answer to the problem of distinguishing between two GI -graphs.

Proposition 1. *Suppose $j_0, j_1, \dots, j_{t-1} \neq 0$ or $n/2$ modulo n , and a is a unit in \mathbb{Z}_n . Then the graph $GI(n; aj_0, aj_1, \dots, aj_{t-1})$ is isomorphic to the graph $GI(n; j_0, j_1, \dots, j_{t-1})$.*

Proof. Since a is coprime to n , the numbers av for $0 \leq v < n$, are all distinct in \mathbb{Z}_n , and so we can label the vertices of $GI(n; aj_0, aj_1, \dots, aj_{t-1})$ as ordered pairs (s, av) for $s \in \mathbb{Z}_t$ and $v \in \mathbb{Z}_n$. Now define a mapping $\varphi : V(GI(n; aj_0, aj_1, \dots, aj_{t-1})) \rightarrow V(GI(n; j_0, j_1, \dots, j_{t-1}))$ by setting $\varphi((s, av)) = (s, v)$ for all $s \in \mathbb{Z}_t$ and all $v \in \mathbb{Z}_n$. This is clearly a bijection, and since a vertex (s, av) in $GI(n; aj_0, aj_1, \dots, aj_{t-1})$ is adjacent to (s', av) for each $s' \in \mathbb{Z}_t \setminus \{s\}$ and to $(s, av \pm aj_s) = (s, a(v \pm j_s))$, it is easy to see that φ is also a graph homomorphism. \square

We may collect the parameters j_s into a multiset J , and then use the abbreviation $GI(n; J)$ for the graph $GI(n; j_0, j_1, \dots, j_{t-1})$. Also we will say that the multiset J is in *canonical form* if it is lexicographically first among all the multisets that give isomorphic copies of $GI(n; J)$ via Proposition 1.

We now list some other properties of GI -graphs. Proposition 3 shows that the spoke edges are easy to recognise when $t > 3$.

Proposition 2. *The graph $GI(n; j_0, j_1, \dots, j_{t-1})$ admits a factorization into a $(t-1)$ -factor nK_t and a 2-factor (namely the spokes and the layers).*

Proposition 3. *An edge of a GI -graph with 4 or more layers is a spoke-edge if and only if it belongs to some clique of size 4.*

Proof. No edge between two vertices in the same layer can lie in a K_4 subgraph, because the subgraph induced on each layer is a union of cycles, and no two spokes between two different layers can have a common vertex. \square

Proposition 4. *Let $d = \gcd(n, j_0, j_1, \dots, j_{t-1})$. Then the graph $GI(n; j_0, j_1, \dots, j_{t-1})$ is a disjoint union of d copies of $GI(n/d; j_0/d, j_1/d, \dots, j_{t-1}/d)$. In particular, the graph $GI(n; j_0, j_1, \dots, j_{t-1})$ is connected if and only if $d = 1$.*

Proof. First observe that the edges of every spoke make up a clique (of order t), so the graph is connected if and only if every two spokes are connected via the layer edges. Now there exists an edge between two spokes S_u and S_v whenever $v - u$ is a multiple of j_s for some s , and hence a path of length 2 between S_u and S_v whenever $v - u$ is a \mathbb{Z}_n -linear combination of some j_s and $j_{s'}$, and so on. Thus S_u and S_v lie in the same connected component of the graph if and only if $v - u$ is expressible (mod n) as a \mathbb{Z} -linear combination of j_0, j_1, \dots, j_{t-1} , say $v - u = cn + c_0j_0 + c_1j_1 + \dots + c_{t-1}j_{t-1}$ for some $c_0, c_1, \dots, c_{t-1} \in \mathbb{Z}$. By Bezout's identity, this occurs if and only if $v - u$ is a multiple of $\gcd(n, j_0, j_1, \dots, j_{t-1}) = d$. It follows that the graph has d components, each containing a set of spokes S_v with $v = u + jd$ for fixed u and variable j , that is, with subscripts differing by multiples of d . Finally, since $(v - u)/d = c(n/d) + c_0(j_0/d) + c_1(j_1/d) + \dots + c_{t-1}(j_{t-1}/d)$, it is easy to see that each component is isomorphic to $GI(n/d; j_0/d, j_1/d, \dots, j_{t-1}/d)$. \square

Finally, note that the restriction of a GI -graph to any proper subset of its layers gives rise to another GI -graph. In particular, if J and K are multisets with $J \subseteq K$, then $GI(n; J)$ is an induced subgraph of $GI(n; K)$.

3 Automorphisms of GI -graphs

In this section, we consider the possible automorphisms of a GI -graph $X = GI(n; J)$, where $J = \{j_0, j_1, \dots, j_{t-1}\}$ is any multiset. If X is disconnected, then since all connected components of X are isomorphic to each other (by Proposition 4), we may simply reduce this to the consideration of automorphisms a connected component of X (and then find the automorphism group using a theorem of Frucht [5], cf. [8]). Hence from now on, we will assume that X is connected.

The set of edges of $X = GI(n; J)$ may be partitioned into spoke edges and layer edges, and we will call this partition of edges the *fundamental edge-partition* of X . We know that the graph induced on the spoke edges is a collection of complete graphs, and that the graph induced on the layer edges is a collection of cycles (with each cycle belonging to a single layer, but with a layer being composed of two or more cycles of the same length $n/\gcd(n, j_s)$ if the corresponding element j_s of J is not a unit mod n).

We will say that an automorphism of X *respects the fundamental edge-partition* if it takes spoke edges to spoke edges, and layer edges to layer edges. Any automorphism of X that does not respect the fundamental edge-partition (and so takes some layer edge to a spoke edge, and some spoke edge to a layer edge) will be called *skew*.

Theorem 5. *Let X be a connected GI -graph with t layers, where $t \geq 2$. If X has a skew automorphism, then either $t = 2$ and X is isomorphic to one of the seven special generalized Petersen graphs $G(4, 1)$, $G(5, 2)$, $G(8, 3)$, $G(10, 2)$, $G(10, 3)$, $G(12, 5)$ and $G(24, 5)$, or $t = 3$ and X is isomorphic to $GI(3; 1, 1, 1)$. Moreover, each of these eight graphs is arc-transitive (and is therefore both vertex-transitive and edge-transitive).*

Proof. First, if $t > 3$ then no layer edge lies in a clique of size t , but every spoke edge does, and therefore no automorphism can map a spoke edge to a layer edge. Thus $t \leq 3$.

Next, suppose $t = 3$, and let φ be an automorphism taking an edge e of some spoke S_v to an edge e' of some layer L_s . Since every edge of a spoke lies in a triangle, namely the spoke itself, it follows that φ must take the whole spoke $S_v = \{(0, v), (1, v), (2, v)\}$ containing e to some triangle containing the layer edge e' , and then the other two edges of the triangle $\{\varphi(0, v), \varphi(1, v), \varphi(2, v)\}$ must be edges from the same layer as e' , namely L_s . It follows that $j_s = n/3$. But then since each of the images $\varphi(0, v), \varphi(1, v), \varphi(2, v)$ lies in two triangles (namely a spoke and a triangle in L_s), each of the vertices $(0, v), (1, v)$ and $(2, v)$ must similarly lie in two triangles, and it follows that all three layers contain a triangle, so $j_0 = j_1 = j_2 = n/3$. In particular, $\gcd(j_0, j_1, j_2) = n/3$, and by connectedness, Proposition 4 implies $n/3 = 1$, so $n = 3$ and $j_0 = j_1 = j_2 = 1$. Thus X is $GI(3; 1, 1, 1)$, which is well-known to be arc-transitive (see [13], for example).

Finally, for the case $t = 2$, everything we need was proved in [6] and [1]. □

Corollary 6. *Every edge-transitive connected GI -graph is isomorphic to one of the eight graphs listed in Theorem 5.*

Hence from now on, we will consider only the automorphisms that respect the fundamental edge-partition. There are three special classes of such automorphisms:

- (1) automorphisms that preserve every layer
- (2) automorphisms that preserve every spoke
- (3) automorphisms that permute both the layers and the spokes non-trivially.

We will consider particular cases of automorphisms of these types below.

Define mappings $\rho : V(X) \rightarrow V(X)$ and $\tau : V(X) \rightarrow V(X)$ given by

$$\rho(s, v) = (s, v + 1) \quad \text{and} \quad \tau(s, v) = (s, -v) \quad \text{for all } s \in \mathbb{Z}_t \text{ and all } v \in \mathbb{Z}_n. \quad (\dagger)$$

Clearly these are automorphisms of X of type (1), permuting the vertices in each layer. Indeed ρ can be viewed as a rotation (of order n), and τ as a reflection (of order 2), and it follows that the automorphism group of X contains a dihedral subgroup of order $2n$, generated by ρ and τ . These $2n$ automorphisms are all of type (1), and all of them respect the fundamental edge-partition of X .

Next, if two of the members of the multiset J are equal, say $j_{s_1} = j_{s_2}$ for $s_1 \neq s_2$, then we have an automorphism λ_{i, s_1, s_2} that exchanges two cycles of layers L_{s_1} and L_{s_2} , but preserves every spoke. These automorphisms are of type (2).

Proposition 7. *Suppose $j_{s_1} = j_{s_2}$ where $s_1 \neq s_2$, and define $d = \gcd(n, j_{s_1}) = \gcd(n, j_{s_2})$. Then for each $i \in \mathbb{Z}_d$, the mapping $\lambda_{i, s_1, s_2} : V(X) \rightarrow V(X)$ given by*

$$\lambda_{i, s_1, s_2}(s, v) = \begin{cases} (s_2, v) & \text{if } s = s_1 \text{ and } v \equiv i \pmod{d} \\ (s_1, v) & \text{if } s = s_2 \text{ and } v \equiv i \pmod{d} \\ (s, v) & \text{otherwise} \end{cases}$$

is an automorphism of X , which respects the fundamental edge-partition, and preserves all layers other than L_{s_1} and L_{s_2} .

Proof. This is obviously a permutation of $V(X)$, preserving adjacency. Moreover, it is also clear that λ_{i, s_1, s_2} preserves every spoke S_v , and exchanges one of the cycles in layer L_{s_1} with the corresponding cycle in layer L_{s_2} , while preserving all other layer cycles. \square

Corollary 8. *Suppose $j_{s_1} = j_{s_2}$ where $s_1 \neq s_2$, and define $d = \gcd(n, j_{s_1}) = \gcd(n, j_{s_2})$. Then the product*

$$\lambda_{s_1, s_2} := \lambda_{0, s_1, s_2} \lambda_{1, s_1, s_2} \cdots \lambda_{d-1, s_1, s_2}$$

is an automorphism of X that respects the fundamental edge-partition, and exchanges layers L_{s_1} and L_{s_2} , while preserving every other layer.

There is another family of automorphisms exchanging layers that exist in some situations; but these automorphism do not preserve spokes, and so they are of type (3):

Proposition 9. *Let a be any unit in \mathbb{Z}_n with the property that $aJ = \{\pm j_0, \pm j_1, \dots, \pm j_{t-1}\}$, and then let $\alpha : \mathbb{Z}_t \rightarrow \mathbb{Z}_t$ be any bijection with the property that $j_{\alpha(s)} = \pm a j_s$ for all $s \in \mathbb{Z}_t$. Then the mapping $\sigma_a : V(X) \rightarrow V(X)$ given by*

$$\sigma_a(s, v) = (\alpha(s), av) \quad \text{for all } s \in \mathbb{Z}_t \text{ and all } v \in \mathbb{Z}_n$$

is an automorphism of X that respects the fundamental edge-partition.

Remarks. Note that the mapping α is not uniquely determined if there exist distinct s_1 and s_2 for which $j_{s_1} = \pm j_{s_2}$, but we can always define the mapping α so that it is a bijection (and satisfies $j_{\alpha(s)} = \pm a j_s$ for all $s \in \mathbb{Z}_t$). Indeed α is uniquely determined if we require that $\alpha(s_1) < \alpha(s_2)$ whenever $s_1 < s_2$ and $j_{s_1} = \pm j_{s_2}$. On the other hand, σ_a is not defined when the condition $aJ = \{\pm j_0, \pm j_1, \dots, \pm j_{t-1}\}$ fails (or equivalently, when $a(J \cup -J) \neq J \cup -J$). Note also that σ_1 is the identity automorphism, while σ_{-1} is the automorphism τ defined earlier, since for $a = -1$ we may take α as the identity permutation and then $\sigma_{-1}(s, v) = (s, -v) = \tau(s, v)$ for every vertex (s, v) .

Proof. First, let b be the multiplicative inverse of a in \mathbb{Z}_n^* . Then for any $(s, v) \in \mathbb{Z}_t \times \mathbb{Z}_n$, we have $\sigma_a(\alpha^{-1}(s), bv) = (\alpha(\alpha^{-1}(s)), abv) = (s, v)$, and therefore σ_a is surjective. Since $V(X)$ is finite, it follows that σ_a is a permutation. Also σ_a preserves edges, indeed it respects the fundamental edge-partition, because it takes each neighbour (s', v) of the vertex (s, v) in the spoke S_v to the neighbour $(\alpha(s'), av)$ of the vertex $(\alpha(s), av)$ in the spoke S_{av} , and takes the two neighbours $(s, v \pm j_s)$ of the vertex (s, v) in the layer L_s to the two neighbours $\sigma_a(s, v \pm j_s) = (\alpha(s), a(v \pm j_s)) = (\alpha(s), av \pm aj_s) = (\alpha(s), av \pm j_{\alpha(s)})$ of the vertex $(\alpha(s), av)$ in the layer $L_{\alpha(s)}$. \square

In the remaining part of this section we will show that if the GI -graph X is connected, then the automorphisms described above and their products give all of the automorphisms of X that respect the fundamental edge-partition.

For this we require two technical Lemmas, the proofs of which are obvious.

Lemma 10. *Let X be a connected GI -graph with at least two layers. Then every automorphism of X that preserves spoke edges must permute the spokes (like blocks of imprimitivity).*

Lemma 11. *Every automorphism of a GI -graph that respects the fundamental edge-partition must permute the layer cycles.*

It will also be helpful to relate the automorphisms of a GI -graph to the automorphisms of the corresponding circulant graph.

Let S be a subset of \mathbb{Z}_n such that $S = -S$ and $0 \notin S$. Then the *circulant graph* $\text{Circ}(n; S)$ is defined as the graph with vertex set \mathbb{Z}_n , such that vertices u and v are adjacent precisely when $u - v \equiv a \pmod n$ for some $a \in S$. Equivalently, this is the Cayley graph for \mathbb{Z}_n given by the subset S . Note that $\text{Circ}(n; S)$ is connected if and only if S additively generates \mathbb{Z}_n , that is, if and only if some linear combination of the members of S is 1 mod n .

Now suppose that $S = \{s_1, \dots, s_c\}$, and that $\Gamma = \text{Circ}(n; S)$ is connected.

For $1 \leq i \leq c$, let $G_{i,1}, G_{i,2}, \dots, G_{i,k_i}$ be the distinct cosets of the cyclic subgroup $G_{i,1} = \langle s_i \rangle$ in $G = \langle S \rangle$. Then we can form a partition $\mathcal{C} = \{C_{ij}\}$ of the edges of Γ , where

$$C_{ij} = \{ \{g, g + s_i\} : g \in G_{i,j} \} \quad \text{for } 1 \leq j \leq k_i \text{ and } 1 \leq i \leq c.$$

Notice that each part C_{ij} of \mathcal{C} consists of precisely the edges of a cycle formed by adding multiples of the single element s_i of S to a member of the coset $G_{i,j}$.

We say that an automorphism φ of Γ *respects the partition* \mathcal{C} if $\varphi(C_{ij}) \in \mathcal{C}$ for every $C_{ij} \in \mathcal{C}$. We have the following, thanks to Joy Morris:

Theorem 12. *Suppose the circulant graph $\Gamma = \text{Circ}(n; S)$ is connected. If ψ is an automorphism of Γ which fixes the vertex 0 and respects the partition $\mathcal{C} = \{C_{ij}\}$, then ψ is induced by some automorphism of \mathbb{Z}_n — that is, there exists a unit $a \in \mathbb{Z}_n$ with the property that $\psi(x) = ax$ for every $x \in \mathbb{Z}_n$ (and in particular, $aS = S$).*

For a proof (by induction on $|S|$), see [16]. To apply it, we associate with our graph $X = GI(n; J)$ the circulant graph $Y = \text{Circ}(n; S \cup -S)$, where S is the underlying set of J . Note that the projection $\eta : V(X) \rightarrow V(Y)$ given by $\eta(s, v) = v$ takes every layer edge $\{(s, v), (s, v + j_s)\}$ of X to the edge $\{v, v + j_s\}$ of Y , and hence gives a graph homomorphism from the subgraph of X induced on layer edges onto the graph Y .

Proposition 13. *Every automorphism of $X = GI(n; J)$ that preserves the set of spoke edges induces an automorphism of $Y = \text{Circ}(n; S \cup -S)$ that respects the partition $\mathcal{C} = \{C_{ij}\}$.*

Proof. Any such automorphism φ induces a permutation on the set of spokes of X , and hence under the above projection η , induces an automorphism of Y , say ψ . Moreover, since φ preserves the layer edges, it must permute the layer cycles among themselves, and it follows that ψ respects the partition $\mathcal{C} = \{C_{ij}\}$. \square

Corollary 14. *Suppose X is connected. Then every automorphism of $X = GI(n; J)$ that respects the fundamental edge-partition of X is expressible as a product of powers of the rotation ρ , the reflection τ , and the automorphisms λ_{i,s_1,s_2} and σ_a defined in Proposition 7 and Proposition 9.*

Proof. First, any such automorphism φ induces a permutation on the set of spokes of X , and so by multiplying by a suitable element of the dihedral group of order $2n$ generated by ρ and τ , we may replace φ by an automorphism φ' that respects the fundamental edge-partition of X , and preserves the spoke S_0 . In particular, φ' induces an automorphism of $Y = \text{Circ}(n; S \cup -S)$ that fixes the vertex 0. By Theorem 12, this automorphism of Y is induced by multiplication by some unit $a \in \mathbb{Z}_n$, and then by multiplying by the inverse of σ_a we may replace φ' by an automorphism φ'' that preserves all of the spokes S_v . Finally, since φ'' preserves all of the spokes and also permutes the layer cycles among themselves, φ'' is expressible as a product of the automorphisms λ_{i,s_1,s_2} defined in Proposition 7. \square

As a special case, we have also the following, for the automorphisms that preserve layers:

Corollary 15. *Suppose X is connected. Then any automorphism of $X = GI(n; J)$ that takes layers to layers is a product of powers of the rotation ρ , the reflection τ , and the automorphisms λ_{s_1,s_2} and σ_a defined in Corollary 8 and Proposition 9.*

4 Automorphism groups of GI -graphs

Now that we know all possible automorphisms of a GI -graph, it is not difficult to determine their number, and construct the automorphism groups in many cases. We will sometimes use $F(n; J)$ to denote the number of automorphisms of $GI(n; J)$, and $A(n; J)$ to denote the automorphism group $GI(n; J)$.

The automorphism group $A(n; J)$ of $GI(n; J)$ always contains a dihedral subgroup of order $2n$, generated by the rotation ρ and the reflection τ , defined in (\dagger) in the previous section (before Proposition 7). Note that the relations $\rho^n = \tau^2 = (\rho\tau)^2 = 1$ hold, with the third of these being equivalent to $\tau\rho\tau = \rho^{-1}$.

We split the consideration of $F(n; J)$ and $A(n; J)$ into four cases, below.

4.1 The disconnected case

Let $d = \gcd(n, J)$. Then $GI(n; J)$ is the disjoint union of d isomorphic copies of $GI(n; J/d)$. This reduces the computation of $\text{Aut}(X)$ to the case of connected GI -graphs. In particular, we have

$$A(n; J) \cong A(n, J/d) \wr \text{Sym}(d)$$

so $\text{Aut}(GI(n; J))$ is the wreath product of $\text{Aut}(GI(n; J/d))$ by the symmetric group $\text{Sym}(d)$ of degree d , and therefore

$$F(n, J) = |\text{Aut}(GI(n; J))| = d! (F(n, J/d))^d.$$

4.2 The edge-transitive case

The eight connected edge-transitive GI -graphs were given in Theorem 5. Seven of them are generalized Petersen graphs, with $J = \{1, k\}$ for some $k \in \mathbb{Z}_n^*$, and their automorphism groups are known — see [6] or [13] for example.

For each of these seven graphs, all of which are cubic, there is an automorphism μ of order 3 that fixes the vertex $(0, 0)$ and induces a 3-cycle on its neighbours $(1, 0)$, $(0, 1)$ and $(0, n-1)$. In particular, this automorphism μ takes the spoke edge $\{(0, 0), (1, 0)\}$ to the layer edge $\{(0, 0), (0, 1)\}$, and its effect on the other vertices is easily determined.

In the cases $(n, k) = (4, 1)$, $(8, 3)$, $(12, 5)$ and $(24, 5)$, where $n \equiv 0 \pmod{4}$ and $k^2 \equiv 1 \pmod{n}$, the three automorphisms ρ , τ and μ generate $A(n; J)$ and satisfy the defining relations

$$\rho^n = \tau^2 = \mu^3 = (\rho\tau)^2 = (\rho\mu)^2 = (\tau\mu)^2 = [\rho^4, \mu] = 1$$

for a group of order $12n$ which we may denote for the time being as $\Gamma(n, k)$, although strictly speaking, the second parameter k is not necessary.

Similarly in the case $(n, k) = (10, 2)$, the three automorphisms ρ , τ and μ generate $A(n; J)$, which has order $12n$, but they satisfy different defining relations, with the relation $[\rho^4, \mu] = 1$ replaced by $\mu\rho^{-1}\mu\rho^2\mu^{-1}\rho^2\tau = 1$. In the other two cases (namely $(n, k) = (5, 2)$ and $(10, 3)$), the automorphisms ρ , τ and μ generate a subgroup of index 2 in $A(n; J)$, which has order $24n$.

In summary, the automorphism groups of the eight connected edge-transitive GI -graphs and their orders can be described as below:

| | |
|--|----------------------|
| $\text{Aut}(GI(4, 1, 1)) \cong \Gamma(4, 1) \cong S_4 \times \mathbb{Z}_2$ | $F(4, 1, 1) = 48$ |
| $\text{Aut}(GI(5, 1, 2)) \cong S_5$ | $F(5, 1, 2) = 120$ |
| $\text{Aut}(GI(8, 1, 3)) \cong \Gamma(8, 3)$ | $F(8, 1, 3) = 96$ |
| $\text{Aut}(GI(10, 1, 2)) \cong A_5 \times \mathbb{Z}_2$ | $F(10, 1, 2) = 120$ |
| $\text{Aut}(GI(10, 1, 3)) \cong S_5 \times \mathbb{Z}_2$ | $F(10, 1, 3) = 240$ |
| $\text{Aut}(GI(12, 1, 5)) \cong \Gamma(12, 5)$ | $F(12, 1, 5) = 144$ |
| $\text{Aut}(GI(24, 1, 5)) \cong \Gamma(24, 5)$ | $F(24, 1, 5) = 288$ |
| $\text{Aut}(GI(3, 1, 1, 1)) \cong (D_6 \times D_6) \rtimes \mathbb{Z}_2$ | $F(3, 1, 1, 1) = 72$ |

See [6] and/or [13] for further details.

4.3 The case where J is a set (with no repetitions)

Suppose J is a set (and not a multiset), in standard form, and let $X = GI(n; J)$. If X is not connected, then sub-section 4.1 applies, while if X is connected and edge-transitive, then sub-section 4.2 applies, so we will suppose that X is connected but not edge-transitive.

Then by Corollary 15, we know that the automorphism group of X is generated by the automorphisms ρ , τ and the set $\{\sigma_a : a \in A\}$, where

$$A = \{a \in \mathbb{Z}_n^* \mid a(J \cup -J) = J \cup -J\}.$$

It is easy to see that A is a subgroup of \mathbb{Z}_n^* . Indeed since σ_1 is trivial, $\sigma_{-1} = \tau$, and $\sigma_a\sigma_b = \sigma_{ab}$ for all $a, b \in A$, the set $S = \{\sigma_a : a \in A\}$ is a subgroup of $\text{Aut}(X)$, isomorphic to A . In particular, S is abelian. It is also easy to see that if composition of functions is read from left to right, and α is the bijection satisfying $j_{\alpha(s)} = \pm a j_s$ for all $s \in \mathbb{Z}_t$, then

$$(\rho\sigma_a)(s, v) = \sigma_a(s, v+1) = (\alpha(s), a(v+1)) = (\alpha(s), av+a) = \rho^a(\alpha(s), av) = (\sigma_a\rho^a)(s, v)$$

for every vertex (s, v) , and so $\rho\sigma_a = \sigma_a\rho^a$ for all $a \in A$. Rearranging, we have $\sigma_a^{-1}\rho\sigma_a = \rho^a$ for all $a \in A$, which shows that every element of S normalizes the cyclic subgroup of order n generated by the rotation ρ . Finally, again since $\tau = \sigma_{-1} \in S$, this implies that the automorphism group of $X = GI(n; J)$ is a semi-direct product:

$$A(n; J) = \langle \{\rho\} \cup S \rangle \cong \langle \rho \rangle \rtimes S \cong C_n \rtimes A, \quad \text{of order } F(n; J) = n|A|.$$

4.4 The general case

In this sub-section we deal with all remaining possibilities, in which J is a multiset with repeated elements, in standard form, and $X = GI(n; J)$ is connected but not edge-transitive. Here we need two new sets of parameters, namely the multiplicity m_j in J of each element j from the underlying set of J (that is, the number of $s \in \mathbb{Z}_t$ for which $j_s = j$), and $d_j = \gcd(n, j)$ for all such j .

Also we need the set B of all $a \in Z_n^*$ with the property that $aJ = \{\pm j_0, \pm j_1, \dots, \pm j_{t-1}\}$. Note that this is always a subgroup of Z_n^* , but is not always the same as the subgroup $A = \{a \in Z_n^* \mid a(J \cup -J) = J \cup -J\}$ that we took in the previous sub-section, since the multiplicities of j and $\pm aj$ in J might not be the same for some $a \in A$, but clearly they must be the same for every $a \in B$.

Now by Corollary 14 we know that the automorphism group of X is generated by the automorphisms ρ and τ , the automorphisms σ_a for $a \in B$ (as defined in Proposition 9), and the automorphisms $\lambda_{i,s,s'}$ (as defined in Proposition 7) that mix cycles.

Just as in the previous case, the set $S = \{\sigma_a : a \in B\}$ is a subgroup of $\text{Aut}(X)$, isomorphic to the subgroup B of Z_n^* . Again also we have $\sigma_a^{-1}\rho\sigma_a = \rho^a$ for all $a \in B$, and so every element of S normalizes the cyclic subgroup of order n generated by the rotation ρ .

Next, for each $j \in J$, define $\Omega_j = \{s \in \mathbb{Z}_t \mid j_s = j\}$, which is a set of size m_j , and for the time being, let $d = d_j = \gcd(n, j)$. Also define $\Omega_{ji} = \{(s, v) \in V(X) \mid s \in \Omega_j, v \equiv i \pmod{d}\}$, for $j \in J$ and $i \in \mathbb{Z}_d$ (where $d = \gcd(n, j)$). Note that $|\Omega_{ji}| = m_j n/d$, because Ω_{ji} is like a strip of vertices across m_j layers of X , containing the n/d vertices of one cycle from each of these layers.

By Proposition 7, for every two distinct s_1, s_2 in Ω_j and every $i \in \mathbb{Z}_d$, there exists an involutory automorphism λ_{i,s_1,s_2} that exchanges one of the d cycles from layer L_{s_1} with the corresponding cycle from layer L_{s_2} , and preserves every spoke. This automorphism induces a transposition on the set of m_j layer cycles containing the vertices of the set Ω_{ji} . If we let the pair $\{s_1, s_2\}$ vary, we get all such transpositions, and hence for fixed $i \in \mathbb{Z}_d$, the automorphisms λ_{i,s_1,s_2} with s_1, s_2 in Ω_j generate a subgroup isomorphic to the symmetric group $\text{Sym}(m_j)$, acting with n/d orbits of length m_j on Ω_{ji} and fixing all other vertices.

Moreover, for any two distinct i_1, i_2 in \mathbb{Z}_d , the elements of T_{i_1} and T_{i_2} move disjoint sets of vertices (namely Ω_{ji_1} and Ω_{ji_2}), and hence commute with each other. Hence the subgroup T_j generated by all of the automorphisms λ_{i,s_1,s_2} with s_1, s_2 in Ω_j is isomorphic to the direct product of d copies of $\text{Sym}(m_j)$, one for each value of i in \mathbb{Z}_d .

Similarly, for any two distinct j, j' in J , the corresponding subgroups T_j and $T_{j'}$ move disjoint sets of vertices (from disjoint sets of layers of X), and hence commute with each other, so the subgroup N generated by the set of all of the automorphisms λ_{i,s_1,s_2} is a direct product $\prod_{j \in J} T_j \cong \prod_{j \in J} (\text{Sym}(m_j))^{d_j}$, of order $\prod_{j \in J} (m_j!)^{d_j}$.

On the other hand, for fixed s_1 and s_2 in Ω_j , then

$$\rho^{-1}\lambda_{i,s_1,s_2}\rho = \lambda_{i+1,s_1,s_2} \quad \text{and} \quad \tau^{-1}\lambda_{i,s_1,s_2}\tau = \lambda_{-i,s_1,s_2} \quad \text{for all } i \in \mathbb{Z}_d,$$

so the automorphisms λ_{i,s_1,s_2} are permuted among themselves in a cycle under conjugation by the rotation ρ , and fixed or interchanged in pairs under conjugation by the reflection τ .

Finally if $a \in B \setminus \{\pm 1\}$, and j' is the element of (the underlying set of) J congruent to $\pm aj \pmod n$, then the automorphism σ_a defined in Proposition 9 takes the layers L_s for $s \in \Omega_j$ to the layers $L_{s'}$ for $s' \in \Omega_{j'}$, and conjugates the subgroup T_j (generated by those λ_{i,s_1,s_2} with s_1, s_2 in Ω_j) to the corresponding subgroup $T_{j'}$. Hence σ_a normalises the subgroup $N = \prod_{j \in J} T_j$.

Thus N is normalised by ρ and τ ($= \sigma_{-1}$) and all the other σ_a , and is therefore normal in $\text{Aut}(X)$. It follows that

$$A(n; J) = \langle N \cup \{\rho\} \cup S \rangle \cong N \rtimes \langle \rho \rangle \rtimes S \cong \prod_{j \in J} (S_{m_j})^{d_j} \rtimes C_n \rtimes B,$$

of order

$$F(n; J) = n |B| \prod_{j \in J} (m_j!)^{d_j},$$

where the products are taken over all j from the underlying set of J , without multiplicities.

4.5 Summary

Combining the results from the four sub-sections above gives an algorithm for computing the automorphism groups of GI -graphs and their automorphism groups in general.

5 Vertex-transitive GI -graphs

In this section we consider further symmetry properties of GI -graphs. By Corollary 6, we know there are only eight different connected edge-transitive GI -graphs $GI(n; J)$ having two or more layers. In particular, there are no such graphs with four or more layers. In contrast, we will show that there are several vertex-transitive GI -graphs, by giving a classification of them.

Note that the graph $GI(n; J)$ will be vertex-transitive if we are able to permute the layers of $GI(n; J)$ transitively among themselves. Now for each non-zero $a \in \mathbb{Z}_n$, consider multiplication of the (multi)set J by a . If this preserves J (as a multiset), then it gives a bijection from J to J , and so by Proposition 9, an automorphism σ_a of $GI(n; J)$, permuting the layers. The graph $GI(n; J)$ will be vertex-transitive if the group generated by all such σ_a acts transitively on the layers.

Theorem 16. *Let J be any subset of \mathbb{Z}_n^* with the two properties that (a) $J \cap -J = \emptyset$, and (b) $J \cup -J$ is a (multiplicative) subgroup of \mathbb{Z}_n^* . Then $GI(n; J)$ is vertex-transitive.*

Proof. Since $J \cup -J$ is a subgroup of \mathbb{Z}_n^* (not containing 0), multiplication by any $a \in J$ gives a bijection from J to J and hence an automorphism σ_a of $GI(n; J)$. Moreover, for any $a, b \in J$ there exists $c \in J$ such that $ac = \pm b$ in \mathbb{Z}_n , and in this case, the automorphism σ_c takes any layer s with $j_s = a$ to a layer s' with $j_{s'} = \pm b$. It follows that the group generated by $\{\sigma_a : a \in J\}$ acts transitively on the layers of $GI(n; J)$, and hence that the group generated by $\{\rho\} \cup \{\sigma_a : a \in J\}$ acts transitively on the vertices of $GI(n; J)$. \square

Corollary 17. *Let A be any subgroup of the multiplicative group \mathbb{Z}_n^* containing an element of $\mathbb{Z}_n \setminus \{\pm 1\}$. If $-1 \in A$, then take $J = A \cap \{1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor\}$ (so that $A = J \cup -J$), while if*

$-1 \notin J$, let $J = A$. Then $GI(n; J)$ is vertex-transitive. Hence for every integer $n > 6$, there exists at least one vertex-transitive GI -graph of the form $GI(n; J)$ for some J with $|J| > 1$.

Note that the above requires $\phi(n) = |\mathbb{Z}_n^*|$ to be at least 4, so that $n > 4$ and $n \neq 6$, in order for there to be at least two layers. A sub-family consists of those for which A is the cyclic subgroup $\{1, r, r^2, \dots, r^{t-1}\}$ generated by the powers of a single unit $r \in \mathbb{Z}_n^* \setminus \{\pm 1\}$. An example is given in Figure 2(b), with $n = 7$ and $r = 2$ (and $2^2 \equiv 4 \equiv -3 \pmod{7}$).

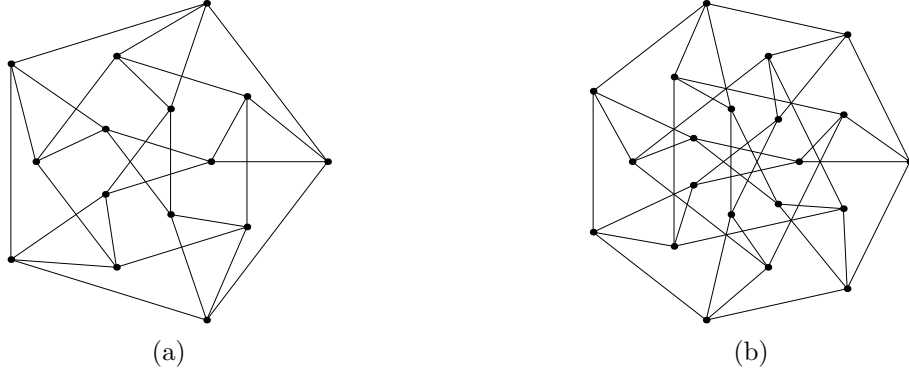


Figure 2: The graph $GI(5; 1, 1, 2)$ in (a) has 5-cycles as its three layers but is not vertex-transitive, while the graph $GI(7; 1, 2, 3)$ in (b) is vertex-transitive and has two edge orbits.

Next, we say that a subset $J = \{j_0, j_1, \dots, j_{t-1}\}$ of \mathbb{Z}_n is *primitive* if $1 \in J$ and $j_i \neq \pm j_k$ whenever $i \neq k$. Also we say that the graph $GI(n; J)$ is *primitive* if J is a primitive subset of \mathbb{Z}_n . Note that any such graph is connected, since $1 \in J$.

Theorem 18. *A primitive GI -graph $GI(n; J)$ is vertex-transitive if and only if either $J \cup -J$ is a (multiplicative) subgroup of \mathbb{Z}_n^* , or $n = 10$ and $J = \{1, 2\}$.*

Proof. First, it was shown in [6] that $GI(10; 1, 2)$ is vertex-transitive. Also by Theorem 16, we know that $GI(n; J)$ is vertex-transitive when $J \cup -J$ is a subgroup of \mathbb{Z}_n^* .

Conversely, suppose that $X = GI(n; J)$ is a primitive vertex-transitive GI -graph, other than $GI(10; 1, 2)$. We have to show that $J \cup -J$ is a subgroup of \mathbb{Z}_n^* .

Since X is primitive, we have $1 \in J$, and without loss of generality we may assume that $j_0 = 1$. By Theorem 5, we know that if X has a skew automorphism, then either $t = 2$ and $(n, j_1) = (4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5)$ or $G(24, 5)$, or $t = 3$ and $(n, j_1, j_2) = (3, 1, 1)$. It is easy to see that $J \cup -J$ is a subgroup of \mathbb{Z}_n^* in all of these cases except $(n, t, j_0, j_1) = (10, 2, 1, 2)$. Hence we may assume that X has no skew automorphism, and therefore every automorphism of X preserves the fundamental edge-partition.

Now because X is vertex-transitive, and the layer L_0 is a single n -cycle, it follows that all the layers of X must be cycles, and so every element of J must be coprime to n , and therefore a unit mod n . In particular, there are no automorphisms that ‘mix’ cycles from different layers. In fact, since J is primitive, $J \cup -J$ contains $2t$ distinct elements, and it follows that X has no automorphisms of the form given in Proposition 7 or Corollary 8.

Hence (by Theorem 12) the only automorphisms that preserve the spoke S_0 are the automorphisms σ_a given in Corollary 15.

But for any $x \in J$ (say $x = j_s$), by vertex-transitivity there exists an automorphism of X that maps the vertex $(0, 0)$ to the vertex $(s, 0)$, and this must be one of the automorphisms

σ_a , where a is a unit in \mathbb{Z}_n and $a(J \cup -J) = J \cup -J$. In particular, since σ_a takes $(0, v)$ to $(\alpha(0), av)$ for all $v \in \mathbb{Z}_n$, we have $\alpha(0) = s$ and therefore $x = j_s = j_{\alpha(0)} = \pm a j_0 = \pm a$, which gives $x(J \cup -J) = \pm a(J \cup -J) = \pm(J \cup -J) = J \cup -J$, for every $x \in J$.

Thus $J \cup -J$ is closed under multiplication, and by finiteness (and the fact that every element of J is a unit mod n), it follows that $J \cup -J$ is a subgroup of \mathbb{Z}_n^* , as required. \square

We will now find some other examples, and show that every vertex-transitive GI -graph has a special form. To do that, we introduce some more notation: we denote by $[k]J$ the concatenation of k copies of the multiset J . Note that this may involve a non-standard ordering of the elements of $[k]J$, but it makes the proofs of some things in this and the next section easier to explain — specifically, Theorem 19 and Lemmas 21 and 22.

Theorem 19. *Let $X = GI(n; J)$ be any connected vertex-transitive GI -graph. Then*

- (a) *If $\text{Aut}(X)$ has a vertex-transitive subgroup that preserves the fundamental edge-partition of X , then $GI(n; [k]J)$ is vertex-transitive for every positive integer k .*
- (b) *All elements in $J \cup -J$ have the same multiplicity, say k_0 , and (so conversely) the graph $X = GI(n; J)$ is isomorphic to $GI(n; [k_0]J_0)$ for some primitive subset J_0 of \mathbb{Z}_n , such that $GI(n; J_0)$ is vertex-transitive.*

Proof. Let $X = GI(n; J) = GI(n; j_0, j_1, \dots, j_{t-1})$, and let $Y = GI(n; [k]J)$.

Note that the vertex-set of Y is $\mathbb{Z}_{kt} \times \mathbb{Z}_n$, and we can write $[k]J = (j_0, j_1, \dots, j_{kt-1})$, where $j_c = j_d$ whenever $c \equiv d \pmod{t}$, and accordingly, we can write each member s of \mathbb{Z}_{kt} in the form $at + b$ where $a \in \mathbb{Z}_k$ and $b \in \mathbb{Z}_t$.

Also note that any permutation f of $\{1, 2, \dots, k\}$ gives rise to a corresponding permutation \tilde{f} of $\{1, 2, \dots, kt\}$, defined by setting $\tilde{f}(at + b) = f(a)t + b$ for all $a \in \mathbb{Z}_k$ and all $b \in \mathbb{Z}_t$, and in fact gives rise to an automorphism $\theta = \theta_f$ of $Y = GI(n; [k]J)$, defined by

$$\theta_f(at + b, v) = (\tilde{f}(at + b), v) = (f(a)t + b, v) \quad \text{for all } a \in \mathbb{Z}_k, b \in \mathbb{Z}_t \text{ and } v \in \mathbb{Z}_n.$$

It is easy to see that θ_f preserves the edges of each spoke S_v , and permutes the layers among themselves. In fact θ_f takes L_{at+b} to $L_{f(a)t+b}$ for all $a \in \mathbb{Z}_k$ and all $b \in \mathbb{Z}_t$, and hence θ_f preserves each of the sets $\{L_s : s \in \mathbb{Z}_{kt} \mid s \equiv b \pmod{t}\}$ for $b \in \mathbb{Z}_t$.

It follows that given any two layers $L_c = \{(c, v) : v \in \mathbb{Z}_n\}$ and $L_d = \{(d, v) : v \in \mathbb{Z}_n\}$ with $c \equiv d \pmod{t}$, there exists an automorphism θ of Y taking L_c to L_d . In particular, since $\text{Aut}(Y)$ is transitive on vertices of each layer (as is the automorphism group of every GI -graph), we find that $\text{Aut}(Y)$ has at most t orbits on vertices of Y .

We can now prove (a), by extending certain automorphisms of X to automorphisms of Y that make it vertex-transitive.

Let ξ be any automorphism of X that respects the fundamental edge-partition. Define a permutation $\pi = \pi_\xi$ of the vertex set of Y by letting

$$\pi(at + b, v) = (at + c, w) \quad \text{whenever } \xi(b, v) = (c, w),$$

for all $a \in \mathbb{Z}_k$, all $b \in \mathbb{Z}_t$, and all $v \in \mathbb{Z}_n$.

If e is a spoke edge of Y , say from $(at + b, v)$ to $(a't + b', v)$, and $(c, w) = \xi(b, v)$, then since ξ takes spoke edges to spoke edges in X , we see that $\xi(b', v) = (c', w)$ for some $c' \in \mathbb{Z}_t$, and so by definition $\pi(a't + b', v) = (a't + c', w)$, which is a neighbour of $(at + c, w)$. Thus π takes the edge e to the spoke edge in Y from $(at + c, w)$ to $(a't + c', w)$.

Similarly, if e is a layer edge of Y , say from $(at + b, v)$ to $(at + b, z)$, with $z = v + j_b$ (since $j_d = j_{d'}$ whenever $d \equiv d' \pmod{t}$), and $(c, w) = \xi(b, v)$, then since ξ permutes the layers of

X , we know that ξ takes the neighbour $(b, z) = (b, v + j_b)$ of (b, v) on the same layer of X as (b, v) to a neighbour of (c, w) on the same layer of X as (c, w) , namely $(c, w \pm j_c)$. Hence by definition, $\pi(at + b, z) = (at + c, w \pm j_c)$, which is a neighbour of $(at + c, w)$ in Y because $j_{at+c} = j_c$. Thus π takes e to a layer edge from $(at + c, w)$ to $(at + c, w \pm j_c)$ in Y .

In particular, since π preserves both the set of all spoke edges of Y and the set of all layer edges of Y , we find that $\pi = \pi_\xi$ is an automorphism of Y .

Moreover, since ξ can be chosen to take any layer of X to any other layer of X , it follows that the subgroup of $\text{Aut}(Y)$ generated by the automorphisms θ_f and π_ξ found above is transitive on layers of Y , and hence Y is vertex-transitive.

Next we prove (b), namely that all elements in $J \cup -J$ have the same multiplicity, say k_0 , and X is isomorphic to $GI(n; [k_0]J_0)$ for some primitive subset J_0 of \mathbb{Z}_n such that $GI(n; J_0)$ is vertex-transitive.

If X is edge-transitive, then by Theorem 5 we have $(n; J) = (4; 1, 1), (5; 1, 2), (8; 1, 3), (10; 1, 2), (10; 1, 3), (12; 1, 5), (24; 1, 5)$ or $(3; 1, 1, 1)$. In the first case, we can take $k_0 = 2$ and $J_0 = \{1\}$, and observe that $GI(n; J_0) = GI(4, 1)$ which is simply a 4-cycle, and vertex-transitive. Similarly, in the last case, we can take $k_0 = 3$ and $J_0 = \{1\}$, and observe that $GI(n; J_0) = GI(3, 1)$ which is a 3-cycle, and vertex-transitive. In all the other six cases, we can take $k_0 = 1$ and $J_0 = J$, and note that $X = GI(n; J)$ itself is vertex-transitive. Thus (b) holds in all eight cases, and so from now on, we may assume that X is not edge-transitive, and hence that every automorphism of X respects the fundamental edge-partition.

This implies that $\text{Aut}(X)$ is transitive on the layers of X , and it follows that all the layer cycles have the same length, so $\gcd(n, j_s) = \gcd(n, j_0)$ for all $s \in \mathbb{Z}_t$. But on the other hand, $X = GI(n; J)$ is connected, so $\gcd(n, j_0, j_1, \dots, j_{t-1}) = 1$. Thus $\gcd(n, j_s) = 1$ for all $s \in \mathbb{Z}_t$.

In particular, there exists $a \in \mathbb{Z}_n^*$ such that $1 = aj_s \in aJ$. Now by Theorem 1, the graph $X = GI(n; J)$ is isomorphic to $GI(n; aJ)$, and therefore we can replace J by aJ , or more simply, suppose that $1 \in J$.

If all the elements of J are the same, then $X = GI(n; J)$ is isomorphic to $GI(n; [t]\{1\})$, and then since the set $\{1\}$ is primitive and $GI(n; 1)$ is simply an n -cycle, again (b) holds.

So now suppose that not all elements of J are the same. For any two distinct $j_i, j_s \in J$, there must be an automorphism σ_a that takes layer L_i to layer L_s , by Corollary 15. In this case $a(J \cup -J) = J \cup -J$, by definition of σ_a , and therefore the multiplicities of j_i and j_s are the same. Hence all elements of $J \cup -J$ have the same multiplicity, say k_0 .

In particular, $J = [k_0]J_0$ where J_0 is the underlying set of J , and X is isomorphic to $GI(n; [k_0]J_0)$. The set J_0 is primitive since it contains 1 and all of its elements are distinct. To finish the proof, all we have to do is show that $GI(n; J_0)$ is vertex-transitive. But that is easy: for any two distinct $j_i, j_s \in J_0$, we know that there exists an automorphism σ_a of X taking layer L_i of X to layer L_s of X , and $a(J \cup -J) = J \cup -J$; it then follows that $a(J_0 \cup -J_0) = J_0 \cup -J_0$, and therefore σ_a induces an automorphism of $GI(n; J_0)$ that takes layer L_i of $GI(n; J_0)$ to layer L_s of $GI(n; J_0)$, as required. \square

Note that the above theorem above applies only to connected GI -graphs. Disconnected vertex-transitive GI -graphs are just disjoint unions of connected vertex-transitive GI -graphs, and can be dealt with accordingly.

We finish this section with observations about the graphs $GI(5; 1, 2)$ and $GI(10; 1, 2)$.

The Petersen graph $GI(5; 1, 2)$ is vertex-transitive, and its automorphism group acts transitively on the two layers; in fact so does a subgroup of order 20 which preserves the set of its ten layer edges. By Theorem 19, it follows that every GI -graph of the form

$GI(5; 1, 2, 1, 2, \dots, 1, 2)$ is vertex-transitive.

On the other hand, the automorphism group of the dodecahedral graph $GI(10; 1, 2)$ has no layer-transitive subgroup preserving the set of layer edges (and the set of spoke edges), and so the above theorem does not apply to it. In fact $GI(10; [k]\{1, 2\})$ is not vertex-transitive for any $k > 1$, because the fact that 2 is not a unit mod 10 implies that the automorphism group has two orbits on layers.

The graph $GI(10; 1, 2)$ is the only such exception, since for every other vertex-transitive GI -graph X , either $\text{Aut}(X)$ itself preserves the fundamental edge-partition, or X is edge-transitive and is then one of the other seven graphs in Theorem 5, and for each of those, the subgroup of $\text{Aut}(X)$ preserving the fundamental edge-partition is layer-transitive.

6 Cayley GI -graphs

In this section we characterise the GI -graphs that are Cayley graphs.

First, a Cayley graph $\text{Cay}(G, S)$ is a graph whose vertices can be labelled with the elements of some group G , and whose edges correspond to multiplication by the elements of some subset S or their inverses. In particular, the edges of $\text{Cay}(G, S)$ may be taken as the pairs $\{g, sg\}$ for all $g \in G$ and all $s \in S$, and then the group G acts naturally as a group of automorphisms of $\text{Cay}(G, S)$ by right multiplication. This action is transitive on vertices, indeed regular on vertices: for any ordered pair (u, v) of vertices, there is a unique element of G taking u to v (namely $g = u^{-1}v$).

Alternatively, a Cayley graph is any (regular) graph X whose automorphism group has a subgroup G that acts regularly on vertices. In that case, any particular vertex can be labelled with the identity element of G , and the subset S can be taken as the set of all $s \in G$ taking that vertex to one of its neighbours.

Note that under both definitions, the Cayley graph is connected if and only if the set S generates the group G . Also note that every Cayley graph is vertex-transitive (by definition), and that every non-trivial element of the subgroup G fixes no vertices of the graph.

Now suppose $X = GI(n; J)$ is a vertex-transitive GI -graph.

We will assume that X is connected, because if it is not, then it is simply a disjoint union of isomorphic copies of a connected smaller example. In particular, by Theorem 19, we know that either J is primitive (and X is one of the graphs given by Theorem 18), or all elements in $J \cup -J$ have the same multiplicity $k_0 > 1$ and then X is isomorphic to $GI(n; [k_0]J_0)$ for some primitive subset J_0 of \mathbb{Z}_n such that $GI(n; J_0)$ is vertex-transitive.

Also we will suppose that X is not $GI(10; 1, 2)$, for reasons related to Theorem 18. In fact, of the seven generalized Petersen graphs among the eight edge-transitive GI -graphs listed in Theorem 5, it is known by the main result of [17] or [14] that $G(4, 1)$, $G(8, 3)$, $G(12, 5)$ and $G(24, 5)$ are Cayley graphs, while $G(5, 2)$, $G(10, 2)$ and $G(10, 3)$ are not. Most of this (and the fact that the eighth edge-transitive graph $GI(3; 1, 1, 1)$ is a Cayley graph) will actually follow from what we prove below.

Consider the case where J is primitive (as we defined in Section 5). In this case, $J \cup -J$ is a subgroup of \mathbb{Z}_n^* under multiplication, and also $|\text{Aut}(X)| = n|J \cup -J| = 2n|J| = 2|V(X)|$.

Hence if G is a subgroup of $\text{Aut}(X)$ that acts regularly on vertices of X , then G is a subgroup of index 2 in $\text{Aut}(X)$. On the other hand, G cannot contain the element τ , since τ is a non-trivial automorphism with fixed points (namely the vertices $(s, 0)$ for all s), and it follows that G must be generated by the rotation ρ and some subgroup of index 2 in $\{\sigma_a : a \in J \cup -J\}$ not containing $\sigma_{-1} = \tau$. The latter has to be of the form $\{\sigma_a : a \in K\}$

for some subgroup K of $J \cup -J$, such that $-1 \notin K$.

Conversely, if J is a set, and K is a subgroup of index 2 in $J \cup -J$ not containing -1 , then the group generated by $\{\sigma_a : a \in K\}$ permutes the layers of X transitively, and so the subgroup generated by $\{\rho\} \cup \{\sigma_a : a \in K\}$ acts regularly on $V(X)$.

Thus we have the following:

Proposition 20. *If $GI(n; J)$ is primitive and $J \cup -J$ is a multiplicative subgroup of \mathbb{Z}_n^* , then $GI(n; J)$ is a Cayley graph if and only if $J \cup -J$ has a subgroup of index 2 that does not contain -1 .*

Note that this gives infinitely many examples of GI -graphs that are Cayley graphs, including those where n is a prime congruent to 3 mod 4 and J is the subgroup of all squares in \mathbb{Z}_n^* . On the other hand, it also gives infinitely many vertex-transitive GI -graphs that are not Cayley graphs, including those where n is a prime congruent to 1 mod 4 and $J \cup -J = \mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\}$.

This theorem also shows that among the six primitive GI -graphs that are edge-transitive, $G(8; 1, 3)$, $G(12; 1, 5)$ and $G(24; 1, 5)$ are Cayley graphs, while $G(5; 1, 2)$ and $G(10; 1, 3)$ are not. (The graph $GI(10; 1, 2)$ is not a Cayley graph, for other reasons.)

Next, consider the more general case, where $X = GI(n; J)$ is connected and vertex-transitive. In this case, by Theorem 19, we know that all elements in $J \cup -J$ have the same multiplicity k_0 , and X is isomorphic to $GI(n; [k_0]J_0)$ for some primitive subset J_0 of \mathbb{Z}_n , such that $GI(n; J_0)$ is vertex-transitive. Also by what we found in Section 5 and sub-section 4.4 of Section 4, we have $d_j := \gcd(n, j) = 1$ for all $j \in J_0$, and therefore

$$|\text{Aut}(X)| = n|J_0 \cup -J_0| \prod_{j \in J_0} (k_0!)^{d_j} = 2n|J_0|(k_0!)^{|J_0|}.$$

We will find the following helpful, and to state it, we will refer to the automorphism ρ of each GI -graph $GI(n; J)$ as its *standard rotation*, and sometimes denote it by ρ_J .

Lemma 21. *If $GI(n; J)$ has a vertex-regular subgroup containing the standard rotation, then so does $GI(n; [k]J)$ for every integer $k > 1$.*

Proof. Let $X = GI(n; J)$ and $Y = GI(n; [k]J)$, and let $\rho (= \rho_J)$ be the standard rotation for X . Also let $\{\rho\} \cup S$ be a generating set for a vertex-regular subgroup of $\text{Aut}(X)$. Note that $\text{Aut}(X)$ is layer-transitive on X , since X is not $GI(10; 1, 2)$. Now by multiplying elements of S by powers of ρ if necessary, we may assume that $\langle S \rangle$ induces a regular permutation group on the set of layers of X . In particular, $\langle S \rangle$ has order J . Next, for each $\xi \in S$, the automorphism π_ξ defined in the proof of Theorem 19 acts fixed-point-freely on $Y = GI(n; [k]J)$, and it follows that the set $\{\pi_\xi : \xi \in S\}$ generates a subgroup of order $|J|$ that permutes the layers of $Y = GI(n; [k]J)$ in $|J|$ blocks of size k . Also if f is the k -cycle $f = (1, 2, \dots, k)$ in $\text{Sym}(k)$, then the automorphism θ_f defined in the proof of Theorem 19 induces a k -cycle on each of those $|J|$ layer-blocks. Finally, θ_f commutes with $\rho_{[k]J}$ and all the π_ξ (for $\xi \in S$), so the subgroup generated by $\rho_{[k]J}$, θ_f and all the π_ξ has order $nk|J|$, and acts regularly on the vertices of Y , as required. \square

Note that this shows, for example, that both of the remaining two edge-transitive GI -graphs $GI(4; 1, 1)$ and $GI(3; 1, 1, 1)$ are Cayley graphs.

Somewhat surprisingly, we also have the following:

Lemma 22. *If J is primitive and both $GI(n; J)$ and $GI(n; [2]J)$ are vertex-transitive, then $GI(n; [2]J)$ is always a Cayley graph, and so is $GI(n; [k]J)$ for every even integer $k > 1$.*

Proof. First, if $GI(n; J)$ is a Cayley graph, then this follows from Lemma 21, so we will assume that $GI(n; J)$ is not a Cayley graph. Also because $GI(n; [2]J)$ is vertex-transitive, we know that $X \neq GI(10; 1, 2)$, and so $J \cup -J$ is a subgroup of \mathbb{Z}_n^* , by Theorem 18. On the other hand, by Proposition 20, we know that $J \cup -J$ has no subgroup of index 2 that excludes -1 . Hence we can write $J \cup -J$ as $U \times W$, where U is a cyclic 2-subgroup containing -1 and of order $q = 2^e$ for some $e > 1$, and W is complementary to U , and of order $2t/q$. Also let u be a generator of U , so that $u^{q/2} = -1$.

Now consider the automorphisms of $Y = GI(n; [2]J)$. For each $a \in J \cup -J = U \times W$, without loss of generality we will choose the associated bijection $\alpha : \mathbb{Z}_{2t} \rightarrow \mathbb{Z}_{2t}$ to be the ‘duplicate’ of the corresponding natural bijection from \mathbb{Z}_t to \mathbb{Z}_t , namely so that α takes s to s' , and $s + t$ to $s' + t$, whenever $j_{s'} = j_{s'+t} = \pm a j_s = \pm a j_{s+t}$ (for $0 \leq s < t$).

For the moment, suppose that W is trivial, so that $U = J \cup -J$. Then the automorphism σ_u is not semi-regular, because the vertex $(0, 0)$ lies in a cycle of length $q/2$ consisting of all $(s, 0)$ with $0 \leq s < t$ and $\pm j_s = u^i$ for some i , while the vertex $(0, 1)$ lies in a cycle of length q consisting of all (s, u^i) such that $0 \leq s < t$ and $\pm j_s = u^i$, for $0 \leq i < q$. Hence in particular, the subgroup generated by ρ and σ_u has order $nq = 2nt$, but cannot be vertex-regular (since the $(q/2)$ th power of σ_u is a non-trivial element with fixed points).

On the other hand, we can multiply σ_u by $\lambda_{0,t}$, which interchanges vertices $(0, v)$ and (t, v) , for all $v \in \mathbb{Z}_n$, and find that $\sigma_u \lambda_{0,t}$ is a semi-regular element of order q , with n/q cycles of length q . (The vertex $(0, 0)$ lies in a cycle of length $q = 2t$ consisting of all $(s', 0)$ with $\pm j_{s'} = u^i$ for some i , while the vertex $(0, 1)$ lies in a cycle of length q consisting of all (s, u^i) such that $0 \leq s < t$ and $\pm j_s = u^i$ for even i , and all $(s + t, u^i)$ such that $0 \leq s < t$ and $\pm j_s = u^i$ for odd i ; the cycles containing the other vertices $(s', 1)$ have a similar form.)

It follows that the subgroup generated by ρ and $\sigma_u \lambda_{0,t}$ has order $nq = 2nt$, and is transitive on vertices, and hence is vertex-regular, so that $GI(n; [2]J)$ is a Cayley graph.

When W is non-trivial, the elements σ_w for all w in W (or simply all w from a generating set for W) induce a regular permutation group on the layers L_s for which $\pm j_s \in W$, and it follows that the subgroup generated by ρ and $\sigma_u \lambda_{0,t}$ and these σ_w acts regularly on the vertices of Y , again making $GI(n; [2]J)$ a Cayley graph.

Finally, for any even integer $k > 2$, we find that $GI(n; [k]J) = GI(n; [k/2][2]J)$ is a Cayley graph, by applying Lemma 21 with $[2]J$ in place of J , and $k/2$ in place of k . \square

On the other hand, the same does not hold when k is odd:

Lemma 23. *If J is primitive and $GI(n; J)$ is vertex-transitive but not a Cayley graph, then $GI(n; [k]J)$ is not a Cayley graph for any odd integer $k > 1$.*

Proof. Assume the contrary, so that $X = GI(n; J)$ is vertex-transitive and not a Cayley graph, but $Y = GI(n; [k]J)$ is a Cayley graph, for some odd k .

Then we know that $X \neq GI(10; 1, 2)$, since Y is vertex-transitive, and so $J \cup -J$ is a subgroup of \mathbb{Z}_n^* , by Theorem 18. On the other hand, since X is not a Cayley graph, Proposition 20 tells us that $J \cup -J$ has no subgroup of index 2 that excludes -1 , and therefore $J \cup -J$ contains an element u of (multiplicative) order $4m$ for some m , with $u^{2m} = -1$. Also by Theorem 5, we know that Y is not edge-transitive, and so $\text{Aut}(Y)$ preserves the fundamental edge-partition of Y , and hence every subgroup of $\text{Aut}(Y)$ permutes the layers of Y among themselves.

Now let R be a vertex-regular subgroup of $\text{Aut}(Y)$, and take $b = u^m$, which has order 4, with $b^2 = -1$ in \mathbb{Z}_n^* . Next, choose i such that $j_i = \pm b$ (noting that such an i must exist because b lies in the subgroup $J \cup -J$). Then by vertex-transitivity of R , there exists some automorphism θ of Y taking the vertex $(0, 0)$ to the vertex $(i, 0)$. Moreover, by our knowledge of the structure of $\text{Aut}(Y)$ from Section 4 and the fact that all of the automorphisms λ_{s_1, s_2} and σ_a preserve the spoke S_0 , it follows that $\theta = w\sigma_b$ or $w\sigma_{-b}$ for some w in the subgroup N generated by the set of all of the automorphisms λ_{s_1, s_2} .

Since R acts regularly on vertices, every non-trivial automorphism in R has to be semi-regular. In particular, θ is semi-regular, as is its square

$$\theta^2 = (w\sigma_{\pm b})^2 = w(\sigma_{\pm b}w\sigma_{\pm b}^{-1})\sigma_{(\pm b)^2} = w'\sigma_{-1} = w'\tau,$$

where $w' = w(\sigma_{\pm b}w\sigma_{\pm b}^{-1}) \in N$. Both w' and τ preserve the spoke S_0 , and therefore so does $w'\tau$, and thus $w'\tau$ acts semi-regularly on S_0 . But also τ fixes every vertex $(s, 0)$ of S_0 , and so w' itself acts semi-regularly on S_0 . Furthermore, since every element of N preserves the set $\{L_0, L_t, \dots, L_{(k-1)t}\}$ of k layers corresponding to the occurrences of 1 in J , it follows that both $w'\tau$ and w' act semi-regularly on the set $K = \{(rt, 0) : 0 \leq r < k\}$.

In particular, cycles of the permutation induced by w' on $K = \{(rt, 0) : 0 \leq r < k\}$ must all have the same length, say ℓ . Note that w' is non-trivial, for otherwise $w'\tau = \tau$, which is not semi-regular on vertices (because it has fixed points), and therefore $\ell > 1$. But also ℓ must divide k , so ℓ is odd.

Now consider any ℓ -cycle of w' on K , say $((s_1, 0), (s_2, 0), \dots, (s_\ell, 0))$. Because τ fixes every vertex of K , this is also a cycle of $w'\tau$, and hence all cycles of $w'\tau$ have length ℓ . Also by definition of the elements generating N (as defined in Proposition 7), we know that $((s_1, 1), (s_1, 1), \dots, (s_\ell, 1))$ must be a cycle of w' . But now the cycle of $w'\tau$ containing the vertex $(s_1, 1)$ is

$$((s_1, 1), (s_2, -1), (s_3, 1), \dots, (s_{\ell-1}, -1), (s_\ell, 1), (s_1, -1), (s_2, 1), (s_3, -1), \dots, (s_\ell, -1)),$$

which has length $2k$, and this contradicts the fact that $w'\tau$ is semi-regular. \square

Putting together Proposition 20 and Lemmas 21 and 22, we have the following:

Theorem 24. *If $X = GI(n; J)$ is connected, then X is a Cayley graph if and only if*

- (a) *J is primitive, and $J \cup -J$ is a multiplicative subgroup of \mathbb{Z}_n^* , with a subgroup of index 2 that does not contain -1 , or*
- (b) *$X = GI(n; J)$ is isomorphic to $GI(n; [k_0]J_0)$ for some primitive subset J_0 of \mathbb{Z}_n and some integer $k_0 > 1$, such that either $GI(n; J_0)$ is a Cayley graph, or k_0 is even and $GI(n; J_0)$ is vertex-transitive but is not the dodecahedral graph $GI(10; 1, 2)$.*

7 Additional remarks

The family of GI -graphs forms a natural generalisation of the Petersen graph. Our initial studies of GI -graphs have shown that this family is indeed very interesting and deserves further consideration. These graphs are also related to circulant graphs [16]. Through that relationship, we were able to solve the puzzle of what appeared to be unstructured automorphisms of GI -graphs, and this enabled us to find their automorphism groups and classify those that are vertex-transitive or Cayley graphs.

Let us mention also the problem of unit-distance drawings of GI -graphs. A graph is a *unit-distance graph* if it can be drawn in the plane such that all of its edges have the

same length. In [10], it was shown that all I -graphs are unit-distance graphs. On the other hand, obviously no GI -graph with four or more layers can be a unit-distance graph, since it contains a K_4 as a subgraph, which itself is not a unit-distance graph. Hence the only open case of interest is the sub-class of GI -graphs having three layers.

For each $k \in \mathbb{Z}_n$, the graph $GI(n; k, k, k)$ is a cartesian product of two cycles and is therefore a unit-distance graph by [9, Theorem 3.4]. We know of only one other connected example that is a unit-distance graph, and it is remarkable.

This is the graph $GI(7; 1, 2, 3)$, which is shown in Figure 3. The vertices can be drawn equidistantly on three concentric circles with radii

$$R_1 = \frac{1}{2 \sin(\pi/7)}, \quad R_2 = \frac{1}{2 \sin(2\pi/7)}, \quad \text{and} \quad R_3 = \frac{1}{2 \sin(3\pi/7)},$$

and the two smaller circles rotated through angles of $\pi/3$ and $-\pi/3$ with respect to the largest circle. One can then verify that all edges have the same length 1.

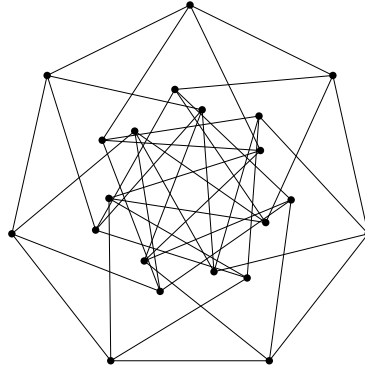


Figure 3: The graph $GI(7; 1, 2, 3)$ as a unit-distance graph.

The graph $GI(7; 1, 2, 3)$ is a Cayley graph for the non-abelian group of order 21, namely $\mathbb{Z}_7 \rtimes_2 \mathbb{Z}_3$, which has presentation $\langle a, b \mid a^7 = b^3 = 1, b^{-1}ab = a^2 \rangle$. Its girth is 3 but it contains no cycles of length 4. This means that its Kronecker cover (see [12]) has girth 6 and is a Levi graph [3] of a self-polar, point- and line-transitive but not flag-transitive combinatorial (21_4) -configuration. The resulting configuration is different from the configuration of Grünbaum and Rigby [7], since the latter configuration is flag-transitive but the one obtained from $GI(7; 1, 2, 3)$ is not.

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